Quantum state transformation by dispersive and absorbing four-port devices

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Abstract

The recently derived input–output relations for the radiation field at a dispersive and absorbing four-port device [T. Gruner and D.-G. Welsch, Phys. Rev. A **54**, 1661 (1996)] are used to derive the unitary transformation that relates the output quantum state to the input quantum state, including radiation and matter and without placing frequency restrictions. It is shown that for each frequency the transformation can be regarded as a well-behaved SU(4) group transformation that can be decomposed into a product of U(2) and SU(2) group transformations. Each of them may be thought of as being realized by a particular lossless four-port device. If for narrow-bandwidth radiation far from the medium resonances the absorption matrix of the four-port device can be disregarded, the well-known SU(2) group transformation for a lossless device is recognized. Explicit formulas for the transformation of Fock-states and coherent states are given.

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I. INTRODUCTION

Four-port devices such as beam splitters are indispensable to optical investigation, and a number of fundamental experiments in quantum optics necessarily require the use of them. The quantum theory of dispersionless and nonabsorbing beam splitters has been well established [1–7]. A beam splitter can be realized by a multislab dielectric plate, which is a dispersive and absorbing device in general. Even if the effects of dispersion and absorption (in a chosen frequency interval) are small, their influence on nonclassical radiation should be considered carefully. On the other hand, in practice multislab dielectric configurations with strongly varying dispersive and absorptive properties, e.g., near optical band gaps, have been of increasing interest, and a description of their action in the quantum domain is desired.

To give a quantum theory of a dispersive and absorbing (linear) four-port device, a Kramers–Kronig consistent quantization scheme of the electromagnetic field in dispersive and absorbing inhomogeneous media is required [8–12]. In particular, quantization of the radiation field within the framework of the phenomenological Maxwell theory (with given complex permittivity in the frequency domain) can be performed using an expansion of the electromagnetic field operators in terms of the Green function of the classical problem and an appropriately chosen infinite set of bosonic basic fields [8]. This quantization scheme, which may be regarded as a generalization of the familiar concepts of mode expansion, applies to any inhomogeneous dielectric matter and is consistent with both the Kramers–Kronig relations and the canonical (equal-time) field commutation relations in QED [11,12].

The formalism has been used in order to derive input–output relations for radiation at a dispersive and absorbing (multilayer) dielectric plate and to express the moments and correlations of the outgoing fields in terms of those of the incoming fields and the (initial) dielectric-matter excitations [9,10,13,15]. Such a (multilayer) dielectric plate may serve as a model for a number of four-port devices, such as beam splitters, mirrors, thin films, interferometers, and optical fibers. The results have been used for studying low-order correlations

in two-photon interference effects [13,14,16].

In this paper we extend the input-output relations for the radiation field at a dispersive and absorbing four-port device to the complete SU(4) group transformations for radiation and matter, and present closed formulas for the transformation of the quantum state as a whole. It is worth noting that the theory applies to optical fields at arbitrary frequencies and bandwidths. In particular for narrow-bandwidth light in which the frequencies are far from medium resonances so that absorption may be disregarded, the well-known results of SU(2) symmetry are recognized. In the general case of nonvanishing absorption, for each frequency the SU(4) group transformation can be given by a product of eight U(2) and SU(2) group transformations, which correspond to an equivalent network of eight lossless four-port devices for radiation and matter.

The paper is organized as follows. In Sec. II the underlying theory is outlined and the basic input-output relations are given. The problem of quantum-state transformation is studied in Sec. III and closed solutions are presented. To illustrate the theory, explicit transformation rules for Fock states and coherent states are presented. A summary and some conclusions are given in Sec. IV.

II. BASIC EQUATIONS

Let us consider two light beams (of fixed polarization) that propagate along the (positive) x_1 and x_2 axes and impinge on a dispersive and absorbing four-port device that gives rise to two outgoing beams propagating along the (positive) y_1 and y_2 axes. Following [13], the operator of the vector potential in each of the four channels of the device can be given by

$$\hat{A}_j(z_j) = \int_0^\infty \mathrm{d}\omega \left[\sqrt{\frac{\hbar \beta_j(\omega)}{4\pi c \omega \epsilon_0 n_j^2(\omega) \mathcal{A}}} \times e^{i\beta_j(\omega)\omega z_j/c} \hat{c}(z_j,\omega) + \mathrm{H.c.} \right] \tag{1}$$

$(j=1,2)$, where

$$n_j(\omega) = \sqrt{\epsilon_j(\omega)} = \beta_j(\omega) + i\,\gamma_j(\omega) \tag{2}$$

is the complex refractive index of the adjacent medium on the jth side of the device ($\mathcal{A}$, plane area of the beam). In Eq. (1), $\hat{c}_j(z_j,\omega)$ stands for the amplitude operators $\hat{a}_j(x_j,\omega)$ and $\hat{b}_j(y_j,\omega)$, respectively, of the incoming and outgoing damped waves at frequency ω. The input-output relations for the amplitude operators can be derived to be

$$\hat{b}_j(\bar{y}_j,\omega) = \sum_{j'=1}^{2} T_{jj'}(\omega)\hat{a}_{j'}(\bar{x}_{j'},\omega) + \sum_{j'=1}^{2} A_{jj'}(\omega)\hat{g}_{j'}(\omega) \tag{3}$$

where it is assumed that the incoming beams enter the device at $x_j=\bar{x}_j$ and the outgoing beams leave the device at $y_j=\bar{y}_j$. The operators $\hat{g}_j(\omega)$ play the role of operator noise sources and describe device excitations. The 2×2 matrices $T_{jj'}(\omega)$ and $A_{jj'}(\omega)$ are the characteristic transformation and absorption matrices of the device. Whereas the $T_{jj'}$ matrix describes the effects of reflection and transmission, the $A_{jj'}$ matrix results from the losses inside the device (for $T_{jj'}(\omega)$ and $A_{jj'}(\omega)$ of a multilayer dielectric slab, see [13]). Finally, the commutation rules for the amplitude operators of the incoming waves and the operators of device excitations are

$$\begin{aligned}[]&[\hat{a}_j(x_j,\omega), \hat{a}^\dagger_{j'}(x'_{j'},\omega')] \\ &\qquad = \delta_{jj'}\delta(\omega-\omega')e^{-\gamma_j(\omega)\,\omega|x_j-x'_{j'}|/c},\end{aligned} \tag{4}$$

$$[\hat{g}_j(\omega), \hat{g}^\dagger_{j'}(\omega')] = \delta_{jj'}\delta(\omega-\omega'), \tag{5}$$

$$[\hat{a}_j(x_j,\omega), \hat{g}_{j'}(\omega')^\dagger] = 0 \tag{6}$$

($x_j \geq \bar{x}_j$). Since the dependence on space of the amplitude operators outside the device is governed by quantum Langevin equations, the input-output relations (3) together with the commutation relations (4) – (6) fully determine the action of the device. The commutation relations (4) – (6) reveal that the amplitude operators of the incoming waves at the entrance plane, $\hat{a}_j(\omega)\equiv\hat{a}_j(\bar{x}_j,\omega)$, and the operators of the device excitations, $\hat{g}_j(\omega)$ are independent bosonic operators. The amplitude operators of the outgoing waves, $\hat{b}_j(\omega)\equiv\hat{b}_j(\bar{y}_j,\omega)$, do not satisfy bosonic commutation relations in general. For given matrices $T_{jj'}(\omega)$ and $A_{jj'}(\omega)$,

their commutation relations can be derived straightforwardly, applying Eq. (3) and using Eqs. (4) – (6).

Let us consider the case when the device is surrounded by vacuum $[n_j(\omega) \to 1]$. In this case the amplitude operators of the incoming and outgoing waves become independent of space and reduce to ordinary bosonic operators. In particular it can be shown that the matrix relation

$$\sum_{k=1}^{2} T_{jk}(\omega) T^{*}_{j'k}(\omega) + \sum_{k=1}^{2} A_{jk}(\omega) A^{*}_{j'k}(\omega) = \delta_{jj'} \tag{7}$$

is valid, which implies the bosonic commutation relation

$$[\hat{b}_j(\omega), \hat{b}^{\dagger}_{j'}(\omega')] = \delta_{jj'}\delta(\omega - \omega'). \tag{8}$$

Note that the relation (7) reflects the fact that when the device is embedded in vacuum, the sum of the probabilities for reflection, transmission, and absorption of a photon is equal to one. When the device is embedded in a medium, the matrix relation (7) and the bosonic commutation relation (8) are not valid in general. From Eqs. (3) – (6) it can be seen that a unitary transformation

$$\hat{b}'_i(\omega) = \sum_{k=1}^{2} X_{ik}(\omega) \hat{b}_k(\omega) \tag{9}$$

$[(X^{-1})_{ik} = X^{*}_{ki}]$ can be introduced such that

$$\left[\lambda_j^{-\frac{1}{2}}(\omega)\hat{b}'_j(\omega), \lambda_{j'}^{-\frac{1}{2}}(\omega)\hat{b}'^{\dagger}_{j'}(\omega')\right] = \delta_{jj'}\delta(\omega - \omega') \tag{10}$$

$(\lambda_j > 0)$. Hence the transformed and scaled operators $\lambda_j^{-\frac{1}{2}}(\omega)\hat{b}'_j(\omega)$ are bosonic operators and the corresponding (scaled and transformed) transformation and absorption matrices satisfy the condition (7).

Without loss of generality we can therefore restrict our attention to a bosonic system and assume that the matrix relation (7) is valid. For notational reasons it is convenient to introduce the definitions

$$\hat{\mathbf{a}}(\omega) = \begin{pmatrix} \hat{a}_1(\omega) \\ \hat{a}_2(\omega) \end{pmatrix}, \tag{11}$$

$$\hat{\mathbf{g}}(\omega) = \begin{pmatrix} \hat{g}_1(\omega) \\ \hat{g}_2(\omega) \end{pmatrix}, \tag{12}$$

$$\hat{\mathbf{b}}(\omega) = \begin{pmatrix} \hat{b}_1(\omega) \\ \hat{b}_2(\omega) \end{pmatrix} \tag{13}$$

and

$$\mathbf{T}(\omega) = \begin{pmatrix} T_{11}(\omega) & T_{12}(\omega) \\ T_{21(\omega)} & T_{22}(\omega) \end{pmatrix}, \tag{14}$$

$$\mathbf{A}(\omega) = \begin{pmatrix} A_{11}(\omega) & A_{12}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) \end{pmatrix}. \tag{15}$$

The input-output relations for radiation at a general four-port device can then be given in the compact form of

$$\hat{\mathbf{b}}(\omega) = \mathbf{T}(\omega)\hat{\mathbf{a}}(\omega) + \mathbf{A}(\omega)\hat{\mathbf{g}}(\omega), \tag{16}$$

with

$$\mathbf{T}(\omega)\mathbf{T}^{+}(\omega) + \mathbf{A}(\omega)\mathbf{A}^{+}(\omega) = \mathbf{I}. \tag{17}$$

III. QUANTUM STATE TRANSFORMATION

The operator input-output relation (16) enables one to calculate arbitrary correlations of the outgoing beams from the correlations of the incoming beams and the device excitations [13]. To obtain the quantum state of the outgoing beams as a whole, the question arises which quantum state transformation corresponds to the operator input-output relation. Let us assume that the incoming fields and the device are prepared in a quantum state described by the density operator $\hat{\varrho}_{\mathrm{in}}$ and that for any frequency the input-output relation (16) corresponds to the unitary operator transformation

$$\hat{\mathbf{b}}(\omega) = \hat{U}^\dagger \hat{\mathbf{a}}(\omega) \hat{U}, \quad \hat{U}^\dagger = \hat{U}^{-1}. \tag{18}$$

The effect of the device can equivalently be described by leaving the photonic operators $\hat{a}_j(\omega)$ unchanged but transforming the input-state density operator $\hat{\varrho}_{\rm in}$ to obtain the output-state density operator $\hat{\varrho}_{\rm out}$ as

$$\hat{\varrho}_{\rm out} = \hat{U} \hat{\varrho}_{\rm in} \hat{U}^\dagger. \tag{19}$$

A. Lossless device

Let us first restrict our attention to a field in a sufficiently small frequency interval of width $\Delta\omega$ in which absorption may be disregarded. For this frequency window the four-port device can be regarded as being lossless, and Eqs. (16) and (17) reduce to

$$\hat{\mathbf{b}}(\omega) = \mathbf{T}(\omega) \hat{\mathbf{a}}(\omega), \tag{20}$$

$$\mathbf{T}(\omega) \mathbf{T}^{+}(\omega) = \mathbf{I}. \tag{21}$$

Here, the elements of the U(2) matrix $\mathbf{T}(\omega)$ are usually given by

$$T_{jj'}(\omega) = \begin{pmatrix} t(\omega) & r(\omega) \\ -r^*(\omega) & t^*(\omega) \end{pmatrix} e^{i\varphi(\omega)}, \tag{22}$$

where $t(\omega)$ and $r(\omega)$, respectively, correspond to the complex transmittance and reflectance of the device at frequency ω,

$$t(\omega) = \cos\theta(\omega)\, e^{i\alpha(\omega)}, \quad r = \sin\theta(\omega)\, e^{i\beta(\omega)}. \tag{23}$$

When the phase shift $\varphi(\omega)$ can be disregarded, then the U(2) group transformation reduces to an SU(2) group transformation. Note that the phase shift can always be included in the input operators by replacing $\hat{\mathbf{a}}(\omega)$ with $\hat{\mathbf{a}}(\omega) e^{i\varphi(\omega)}$, so that $\mathbf{T}(\omega)$ becomes an SU(2) group matrix. The unitary exponential operator $\hat{U}$ in Eqs. (18) and (19) can easily be found by extending the formalism of lossless beam-splitter transformation [1,5] to multi-mode fields:

$$\hat{U} = \exp\left[-i\int_{\Delta\omega} \mathrm{d}\omega \, (\hat{\mathbf{a}}^\dagger(\omega))^T \mathbf{V}(\omega)\hat{\mathbf{a}}(\omega)\right] \tag{24}$$

(the superscript T introduces transposition), where the 2×2 Hermitian matrix $\mathbf{V}(\omega)$ is related to the SU(2) matrix $\mathbf{T}(\omega)$ in Eq. (20) as

$$\exp[-i\mathbf{V}(\omega)] = \mathbf{T}(\omega). \tag{25}$$

The operator $\hat{U}$ can be factored in different ways, e.g.,

$$\begin{aligned}\hat{U} = \exp&\left\{i\int_{\Delta\omega} \mathrm{d}\omega \, \varphi(\omega)[\hat{a}_1^\dagger(\omega)\hat{a}_1(\omega) + \hat{a}_2^\dagger(\omega)\hat{a}_2(\omega)]\right\} \\ &\times \exp\left[\int_{\Delta\omega} \mathrm{d}\omega \, \ln t(\omega) \, \hat{a}_1^\dagger(\omega)\hat{a}_1(\omega)\right] \\ &\times \exp\left[-\int_{\Delta\omega} \mathrm{d}\omega \, r^*(\omega) \, \hat{a}_2^\dagger(\omega)\hat{a}_1(\omega)\right] \\ &\times \exp\left[\int_{\Delta\omega} \mathrm{d}\omega \, r(\omega) \, \hat{a}_1^\dagger(\omega)\hat{a}_2(\omega)\right] \\ &\times \exp\left[-\int_{\Delta\omega} \mathrm{d}\omega \, \ln t(\omega) \, \hat{a}_2^\dagger(\omega)\hat{a}_2(\omega)\right].\end{aligned} \tag{26}$$

B. Dispersive and absorbing device

1. Transformation law

In order to apply the input–output relation (16) [together with Eq. (17)], we first extend it to a U(4) group transformation. For this purpose we combine the two-dimensional vectors $\hat{\mathbf{a}}(\omega)$ and $\hat{\mathbf{g}}(\omega)$ to obtain a four-dimensional input vector

$$\hat{\boldsymbol{\alpha}}(\omega) = \begin{pmatrix} \hat{\mathbf{a}}(\omega) \\ \hat{\mathbf{g}}(\omega) \end{pmatrix} \tag{27}$$

and supply the two-dimensional vector $\hat{\mathbf{b}}(\omega)$ with some other two-dimensional vector $\hat{\mathbf{h}}(\omega)$ to obtain a four-dimensional output vector

$$\hat{\boldsymbol{\beta}}(\omega) = \begin{pmatrix} \hat{\mathbf{b}}(\omega) \\ \hat{\mathbf{h}}(\omega) \end{pmatrix}. \tag{28}$$

Now we relate the four-dimensional vector $\hat{\boldsymbol{\beta}}(\omega)$ to the four-dimensional vector $\hat{\boldsymbol{\alpha}}(\omega)$ as

$$\hat{\boldsymbol{\beta}}(\omega) = \boldsymbol{\Lambda}(\omega)\hat{\boldsymbol{\alpha}}(\omega), \tag{29}$$

$$\boldsymbol{\Lambda}(\omega)\boldsymbol{\Lambda}^{+}(\omega) = \boldsymbol{I}, \tag{30}$$

where the 4×4 unitary matrix $\boldsymbol{\Lambda}(\omega)$ is chosen such that the input–output relation (16) between $\hat{\mathbf{b}}(\omega)$ and $\hat{\mathbf{a}}(\omega)$ is preserved. The matrix $\boldsymbol{\Lambda}(\omega)$ can be expressed in terms of 2×2 matrices as (App. A)

$$\boldsymbol{\Lambda}(\omega) = \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{A}(\omega) \\ -\mathbf{S}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{C}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix}, \tag{31}$$

where

$$\mathbf{C}(\omega) = \sqrt{\mathbf{T}(\omega)\mathbf{T}^{+}(\omega)} \tag{32}$$

and

$$\mathbf{S}(\omega) = \sqrt{\mathbf{A}(\omega)\mathbf{A}^{+}(\omega)} \tag{33}$$

are commuting positive Hermitian matrices, and

$$\mathbf{C}(\omega)^2 + \mathbf{S}(\omega)^2 = \mathbf{I}. \tag{34}$$

In Eq. (31) the unitary matrix $\mathbf{D}(\omega)$ that appears in Eq. (A11) in App. A has been omitted, since it corresponds to an irrelevant change of the device variables $\hat{\mathbf{h}}(\omega)$, as it can be seen from the second line in the large brackets in Eq. (31). Note that after separation of phase factors $e^{i\varphi(\omega)}$ and $e^{i\psi(\omega)}$, respectively, from the matrices $\mathbf{T}(\omega)$ and $\mathbf{A}(\omega)$ and inclusion of them in the operators $\hat{\mathbf{a}}(\omega)$ and $\hat{\mathbf{g}}(\omega)$ the matrix $\boldsymbol{\Lambda}(\omega)$ can be regarded as an SU(4) matrix.

The input-output relation (29) can be expressed in terms of a unitary operator transformation

$$\hat{\boldsymbol{\beta}}(\omega) = \hat{U}^{\dagger}\hat{\boldsymbol{\alpha}}(\omega)\hat{U}, \tag{35}$$

where the unitary operator $\hat{U}$ is given by

$$\hat{U} = \exp\left[-i\int_0^\infty \mathrm{d}\omega \, (\hat{\boldsymbol{\alpha}}^\dagger(\omega))^T \boldsymbol{\Phi}(\omega)\hat{\boldsymbol{\alpha}}(\omega)\right]. \tag{36}$$

Here, $\boldsymbol{\Phi}(\omega)$ is a 4×4 Hermitian matrix which is related to the SU(4) matrix $\boldsymbol{\Lambda}(\omega)$ by

$$\exp[-i\boldsymbol{\Phi}(\omega)] = \boldsymbol{\Lambda}(\omega). \tag{37}$$

Note that for narrow-bandwidth radiation far from medium resonances the ω integral in Eq. (36) can be restricted to a small interval in which absorption may be disregarded, $\mathbf{A}(\omega) \approx \mathbf{0}$, and hence

$$\boldsymbol{\Lambda}(\omega) \approx \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{38}$$

$$\boldsymbol{\Phi}(\omega) \approx \begin{pmatrix} \mathbf{V}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{39}$$

In this case Eq. (36) approximately reduces to Eq. (24) and the SU(2) group transformation for a lossless device is recognized.

Obviously, the first and the second line in the vector equation (35) correspond to the input–output relation (18). Application of Eq. (18) then yields the output quantum state $\hat{\varrho}_{\mathrm{out}}$, from which the quantum state of the outgoing radiation field, $\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})}$, can be derived,

$$\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})} = \mathrm{Tr}^{(\mathrm{D})}\{\hat{\varrho}_{\mathrm{out}}\} = \mathrm{Tr}^{(\mathrm{D})}\{\hat{U}\hat{\varrho}_{\mathrm{in}}\hat{U}^\dagger\}, \tag{40}$$

where $\mathrm{Tr}^{(\mathrm{D})}$ means the trace with respect to the device. The input density operator $\hat{\varrho}_{\mathrm{in}}$ is an operator functional of $\hat{\boldsymbol{\alpha}}(\omega)$ and $\hat{\boldsymbol{\alpha}}^\dagger(\omega)$,

$$\hat{\varrho}_{\mathrm{in}} = \hat{\varrho}_{\mathrm{in}}[\hat{\boldsymbol{\alpha}}(\omega), \hat{\boldsymbol{\alpha}}^\dagger(\omega)], \tag{41}$$

and hence the transformed density operator $\hat{\varrho}_{\mathrm{out}}$ can be given by

$$\hat{\varrho}_{\mathrm{out}} = \hat{\varrho}_{\mathrm{in}}[\hat{U}\hat{\boldsymbol{\alpha}}(\omega)\hat{U}^\dagger, \hat{U}\hat{\boldsymbol{\alpha}}^\dagger(\omega)\hat{U}^\dagger]. \tag{42}$$

Recalling Eqs. (29) and (35), we see that

$$\hat{U}\hat{\boldsymbol{\alpha}}(\omega)\hat{U}^{\dagger} = \boldsymbol{\Lambda}^{+}(\omega)\hat{\boldsymbol{\alpha}}(\omega), \tag{43}$$

$$\hat{U}\hat{\boldsymbol{\alpha}}^{\dagger}(\omega)\hat{U}^{\dagger} = \boldsymbol{\Lambda}^{T}(\omega)\hat{\boldsymbol{\alpha}}^{\dagger}(\omega). \tag{44}$$

Combining Eqs. (40) – (44), we derive

$$\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})} = \mathrm{Tr}^{(\mathrm{D})}\Big\{\hat{\varrho}_{\mathrm{in}}[\boldsymbol{\Lambda}^{+}(\omega)\hat{\boldsymbol{\alpha}}(\omega), \boldsymbol{\Lambda}^{T}(\omega)\hat{\boldsymbol{\alpha}}^{\dagger}(\omega)]\Big\}\,. \tag{45}$$

2. Relation to U(2) and SU(2) group transformations

As shown in App. B, the U(4) group transformation defined by the matrix $\boldsymbol{\Lambda}$ given in Eq. (A11) is equivalent to five U(2) group transformations. That is to say, for chosen frequency the action of an absorbing four-port device formally corresponds to the combined action of five lossless four-port devices in general (for the factorization of an U(N) matrix into U(2) matrices, see also [17]). When the irrelevant matrix $\mathbf{D}(\omega)$ in Eq. (A11) is set equal to the unit matrix $\mathbf{I}$, then Eq. (A11) reduces to Eq. (31) and the action of the absorbing device corresponds, for chosen frequency, reduces to the combined action of eight lossless devices in general. In this case the unitary operator (36) can be factored into a product of unitary operators of the type given in Eq. (24) for a lossless device,

$$\hat{U}[\mathbf{M};\hat{\mathbf{q}}] \equiv \exp\!\Big[-i\int_0^\infty \mathrm{d}\omega\,\big(\hat{\mathbf{q}}^{\dagger}(\omega)\big)^T\mathbf{W}(\omega)\hat{\mathbf{q}}(\omega)\Big]\,. \tag{46}$$

Here, $\mathbf{W}(\omega)$ is a 2×2 Hermitian matrix that is related to a U(2) group transformation matrix $\mathbf{M}(\omega)$ as

$$\exp[-i\mathbf{W}(\omega)] = \mathbf{M}(\omega), \tag{47}$$

and $\hat{\mathbf{q}}(\omega)$ is a vector whose two components are bosonic operators. Note that for narrow-bandwidth radiation far from medium resonances Eq. (46) [together with Eq. (47)] corresponds to Eq. (24) [together with Eq. (25)], with $\mathbf{M}(\omega) = \mathbf{T}(\omega)$, $\mathbf{W}(\omega) = \mathbf{V}(\omega)$, and $\hat{\mathbf{q}}(\omega) = \hat{\mathbf{a}}(\omega)$. As shown in App. B, the unitary operator given in Eq. (36),

$$U \equiv \hat{U}[\mathbf{\Lambda};\hat{\boldsymbol{\alpha}}] = \exp\biggl[-i\int_0^\infty \mathrm{d}\omega\,(\hat{\boldsymbol{\alpha}}^\dagger)^T(\omega)\mathbf{\Phi}(\omega)\hat{\boldsymbol{\alpha}}(\omega)\biggr]\,, \tag{48}$$

can be decomposed into a product of operators $U[\mathbf{M};\hat{\mathbf{q}}]$ as follows:

$$\begin{aligned}\hat{U}[\mathbf{\Lambda};\hat{\boldsymbol{\alpha}}] &= \hat{U}[\mathbf{C}{+}i\mathbf{S};(i\hat{\mathbf{a}}{+}\hat{\mathbf{g}})/\sqrt{2}]\\ &\times\hat{U}[\mathbf{C}{-}i\mathbf{S};(\hat{\mathbf{a}}{+}i\hat{\mathbf{g}})/\sqrt{2}]\,\hat{U}[\mathbf{S}^{-1}\mathbf{A};\hat{\mathbf{g}}]\,\hat{U}[\mathbf{C}^{-1}\mathbf{T};\hat{\mathbf{a}}]\end{aligned} \tag{49}$$

[cf. Eqs. (B13), (B18), and (B24)], and decomposition of $\hat{U}[\mathbf{C}{-}i\mathbf{S};(\hat{\mathbf{a}}{+}i\hat{\mathbf{g}})/\sqrt{2}]$ and $\hat{U}[\mathbf{C}{+}i\mathbf{S};(i\hat{\mathbf{a}}{+}\hat{\mathbf{g}})/\sqrt{2}]$ eventually yields

$$\begin{aligned}\hat{U}[\mathbf{\Lambda};\hat{\boldsymbol{\alpha}}] = \hat{U}^\dagger[\mathbf{P};\hat{\mathbf{d}}_2]\,&\hat{U}^\dagger[\mathbf{P};\hat{\mathbf{d}}_1]\\ &\times\hat{U}[\mathbf{C}{+}i\mathbf{S};\hat{\mathbf{g}}]\,\hat{U}[\mathbf{C}{-}i\mathbf{S};\hat{\mathbf{a}}]\\ &\times\hat{U}[\mathbf{P};\hat{\mathbf{d}}_2]\,\hat{U}[\mathbf{P};\hat{\mathbf{d}}_1]\\ &\times\hat{U}[\mathbf{S}^{-1}\mathbf{A};\hat{\mathbf{g}}]\,\hat{U}[\mathbf{C}^{-1}\mathbf{T};\hat{\mathbf{a}}],\end{aligned} \tag{50}$$

where

$$\hat{\mathbf{d}}_j(\omega) = \begin{pmatrix}\hat{a}_j(\omega)\\ \hat{g}_j(\omega)\end{pmatrix} \tag{51}$$

$(j=1,2)$ and

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \tag{52}$$

[cf. Eqs. (B26) and (B27)]. It should be pointed out that when $\mathbf{\Lambda}(\boldsymbol{\omega})$ is an SU(4) group transformation, then the matrices $\mathbf{P}$, $\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega)$, and $\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega)$ correspond to SU(2) group transformations. The matrices $\mathbf{C}(\omega)+i\mathbf{S}(\omega)$ and $\mathbf{C}(\omega)-i\mathbf{S}(\omega)$ correspond to U(2) group transformations in general, i.e., SU(2) transformations and additional phase shifts. Needless to say that each of the operators $\hat{U}[\mathbf{M};\hat{\mathbf{q}}]$ on the right-hand side in Eq. (50) can be further factored, e.g., according to Eq. (26).

3. Discretization

In quantum optics radiation fields are frequently described in terms of discrete modes. Here we restrict attention to (quasi-)monochromatic discrete modes. For this purpose we subdivide the frequency axis into sufficiently small intervals Δ_m with midfrequencies ω_m and define the bosonic input operators

$$\hat{\boldsymbol{\alpha}}_m = \frac{1}{\sqrt{\Delta_m}} \int_{\Delta_m} \mathrm{d}\omega \, \hat{\boldsymbol{\alpha}}(\omega) \tag{53}$$

and the bosonic output operators $\hat{\boldsymbol{\beta}}_m$ accordingly. The operator input–output relation (29) then reads as

$$\hat{\boldsymbol{\beta}}_m = \boldsymbol{\Lambda}_m \hat{\boldsymbol{\alpha}}_m, \tag{54}$$

$[\boldsymbol{\Lambda}_m = \boldsymbol{\Lambda}(\omega_m)]$, which can be rewritten as

$$\hat{\boldsymbol{\beta}}_m = \hat{U}^\dagger \hat{\boldsymbol{\alpha}}_m \hat{U} = \hat{U}_m^\dagger \hat{\boldsymbol{\alpha}}_m \hat{U}_m \,, \tag{55}$$

where [in place of (36)]

$$\hat{U} = \prod_m \hat{U}_m \,, \tag{56}$$

with

$$\hat{U}_m = \exp\Big[-i(\hat{\boldsymbol{\alpha}}_m^\dagger)^T \boldsymbol{\Phi}_m \hat{\boldsymbol{\alpha}}_m\Big] \,, \tag{57}$$

and according to Eq. (37), the 4×4 Hermitian matrix $\boldsymbol{\Phi_m}$ is related to the SU(4) matrix $\boldsymbol{\Lambda_m}$ as

$$\exp[-i\boldsymbol{\Phi_m}] = \boldsymbol{\Lambda_m} \,. \tag{58}$$

The input density operator $\hat{\varrho}_{\mathrm{in}}$ is now an operator function of $\hat{\boldsymbol{\alpha}}_m$ and $\hat{\boldsymbol{\alpha}}_m^\dagger$, and according to Eq. (45), the density operator of the outgoing radiation field can be given by

$$\hat{\varrho}_{\mathrm{out}}^{(\mathrm{F})} = \mathrm{Tr}^{(\mathrm{D})} \Big\{ \hat{\varrho}_{\mathrm{in}} [\boldsymbol{\Lambda}_m^{+} \hat{\boldsymbol{\alpha}}_m, \boldsymbol{\Lambda}_m^{T} \hat{\boldsymbol{\alpha}}_m^\dagger] \Big\} \,. \tag{59}$$

In close analogy to Eqs. (49) and (50), each SU(4)-group transformation operator $\hat{U}_m$ can be decomposed into a product of U(2)- and SU(2)-group transformation operators $\hat{U}_m[\mathbf{M}_m;\hat{\mathbf{q}}_m]$,

$$\begin{aligned}\hat{U}_m[\mathbf{\Lambda}_{\boldsymbol{m}};\hat{\boldsymbol{\alpha}}_m] &= \hat{U}_m[\mathbf{C}_m+i\mathbf{S}_m;(i\hat{\mathbf{a}}_m+\hat{\mathbf{g}}_m)/\sqrt{2}]\\ &\quad\times\hat{U}_m[\mathbf{C}_m-i\mathbf{S}_m;(\hat{\mathbf{a}}_m+i\hat{\mathbf{g}}_m)/\sqrt{2}]\\ &\quad\times\hat{U}[\mathbf{S}_m^{-1}\mathbf{A}_m;\hat{\mathbf{g}}_m]\,\hat{U}[\mathbf{C}_m^{-1}\mathbf{T}_m;\hat{\mathbf{a}}_m]\\ &= \hat{U}_m^\dagger[\mathbf{P};\hat{\mathbf{d}}_{m2}]\,\hat{U}_m^\dagger[\mathbf{P};\hat{\mathbf{d}}_{m1}]\\ &\quad\times\hat{U}_m[\mathbf{C}_m+i\mathbf{S}_m;\hat{\mathbf{g}}_m]\,\hat{U}_m[\mathbf{C}_m-i\mathbf{S}_m;\hat{\mathbf{a}}_m]\\ &\quad\times\hat{U}[\mathbf{P};\hat{\mathbf{d}}_{m2}]\,\hat{U}_m[\mathbf{P};\hat{\mathbf{d}}_{m1}]\\ &\quad\times\hat{U}_m[\mathbf{S}_m^{-1}\mathbf{A}_m;\hat{\mathbf{g}}_m]\,\hat{U}_m[\mathbf{C}_m^{-1}\mathbf{T}_m;\hat{\mathbf{a}}_m]\,. \end{aligned}\tag{60}$$

Here $\hat{U}_m[\mathbf{M}_m;\hat{\mathbf{q}}_m]$ is given by

$$\hat{U}_m[\mathbf{M}_m;\hat{\mathbf{q}}_m] \equiv \exp\Big[-i(\hat{\mathbf{q}}_m^\dagger)^T\mathbf{W}_m\hat{\mathbf{q}}_m\Big]\,, \tag{61}$$

where

$$\exp[-i\mathbf{W}_m] = \mathbf{M}_m \tag{62}$$

[cf. Eqs. (46) and (47)]. Recalling the definition of discrete operators, Eq. (53), application of Eq. (26) to $\hat{U}_m[\mathbf{M}_m;\hat{\mathbf{q}}_m]$ for further factorization is straightforward.

4. Transformation of Fock-states and coherent states

To illustrate the theory, let us consider the transformation of Fock states and coherent states as fundamental basis states for quantum state representation. For the sake of transparency, we will restrict attention to single-mode states at some chosen frequency, so that the subscript m can be omitted. The results can easily be extended to multimode fields by taking the direct product of single-mode states. Let be

$$\hat{\varrho}_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|, \tag{63}$$

$$|\psi_{\rm in}\rangle = |n_1; n_2, n_3; n_4\rangle = \prod_{\nu=1}^{4} \frac{\hat{\alpha}_\nu^{\dagger\, n_\nu}}{\sqrt{n_\nu!}}\, |0\rangle, \tag{64}$$

the density operator of the system in the case when n_1 and n_2 photons impinge on the device that is excited in Fock states with n_3 and n_4 quanta. From Eq. (59) we then obtain

$$\hat{\varrho}_{\rm out}^{(\rm F)} = {\rm Tr}^{(\rm D)}\{|\psi_{\rm out}\rangle\langle\psi_{\rm out}|\}\,, \tag{65}$$

with

$$|\psi_{\rm out}\rangle = \prod_{\nu=1}^{4} \frac{1}{\sqrt{n_\nu!}} \left(\sum_{\mu=1}^{4} \Lambda_{\mu\nu}\hat{\alpha}_\mu^\dagger\right)^{n_\nu} |0\rangle. \tag{66}$$

We use the decomposition

$$\left(\sum_{\mu=1}^{4} \Lambda_{\mu\nu}\hat{\alpha}_\mu^\dagger\right)^{n_\nu} = \sum_{\{k_{\nu\mu}\}} \prod_{\mu=1}^{4} \frac{n_\nu!}{k_{\nu\mu}!} \left(\Lambda_{\mu\nu}\hat{\alpha}_\mu^\dagger\right)^{k_{\nu\mu}}, \tag{67}$$

where the (non-negative) integers $k_{\nu\mu}$ satisfy the condition

$$\sum_{\mu=1}^{4} k_{\nu\mu} = n_\nu\,, \tag{68}$$

and arrive at

$$|\psi_{\rm out}\rangle = \sum_{\{k_\mu\}} C_{k_1,k_2,k_3,k_4} |k_1; k_2; k_3; k_4\rangle \tag{69}$$

where

$$\begin{aligned} &C_{k_1,k_2,k_3,k_4} \\ &\quad = \left(\prod_{\nu=1}^{4}\sqrt{n_\nu!}\right)\left(\prod_{\mu=1}^{4}\sqrt{k_\mu!}\right) \sum_{\{k_{\nu\mu}\}} \prod_{\mu,\nu=1}^{4} \frac{\Lambda_{\mu\nu}^{k_{\nu\mu}}}{k_{\nu\mu}!}\,, \end{aligned} \tag{70}$$

the $k_{\nu\mu}$ satisfying the conditions

$$\sum_{\mu=1}^{4} k_{\nu\mu} = n_\nu\,, \quad \sum_{\nu=1}^{4} k_{\nu\mu} = k_\mu\,. \tag{71}$$

Using Eqs. (65), (69), and (70), the quantum state of the outgoing radiation-field modes can easily be obtained,

$$\varrho_{\rm out}^{(\rm F)} = \sum_{k_1,k_2} \sum_{k'_1,k'_2} D_{k_1,k_2,k'_1,k'_2}\, |k_1; k_2\rangle\langle k'_1; k'_2|, \tag{72}$$

$$D_{k_1,k_2,k_1',k_2'} = \sum_{k_3,k_4} C_{k_1,k_2,k_3,k_4} C^*_{k_1',k_2',k_3,k_4} \,. \tag{73}$$

The density operator of the outgoing field, $\varrho^{(\mathrm{F})}_{\mathrm{out}}$, can be represented in another way which more clearly shows the influence on the outgoing field state of the device. Let us define linear combinations

$$\hat{x}^\dagger_\nu = \sum_{i=1}^{2} \Lambda_{i\nu} \hat{a}^\dagger_i \tag{74}$$

and

$$\hat{y}^\dagger_\nu = \sum_{i=1}^{2} \Lambda_{2+i\,\nu} \hat{g}^\dagger_i \tag{75}$$

of the photonic and device operators, respectively. Making in Eq. (66) the insertion

$$\sum_{\mu=1}^{4} \Lambda_{\mu\nu} \hat{\alpha}^\dagger_\mu = \hat{x}^\dagger_\nu + \hat{y}^\dagger_\nu, \tag{76}$$

Eq. (65) then reads as

$$\varrho^{(\mathrm{F})}_{\mathrm{out}} = \sum_{\{p_\mu\}\{q_\nu\}} Y_{\{p_\mu\}\{q_\nu\}} \prod_{\mu=1}^{4} \hat{x}^{\dagger p_\mu}_\mu |0^{(F)}\rangle\langle 0^{(F)}| \prod_{\nu=1}^{4} \hat{x}^{q_\nu}_\nu \,, \tag{77}$$

where

$$\begin{aligned} Y_{\{p_\mu\}\{q_\nu\}} &= \left[\prod_{\nu=1}^{4} \frac{1}{\sqrt{n_\nu!}} \binom{n_\nu}{q_\nu}\right] \left[\prod_{\mu=1}^{4} \frac{1}{\sqrt{n_\mu!}} \binom{n_\mu}{p_\mu}\right] \\ &\times \langle 0^{(D)}| \left(\prod_{\nu=1}^{4} \hat{y}^{n_\nu - q_\nu}_\nu\right) \left(\prod_{\mu=1}^{4} \hat{y}^{\dagger n_\mu - p_\mu}_\mu\right) |0^{(D)}\rangle, \end{aligned} \tag{78}$$

and $|0^{(F)}\rangle$ ($|0^{(D)}\rangle$) is the ground state of the field (device). The device vacuum expectation value in Eq. (78) can be calculated by moving the operators $\hat{y}_\nu$ from left to right and employing the commutation relations between $\hat{y}_\nu$ and $\hat{y}^\dagger_\mu$. Then we find, that the coefficients $Y_{\{p_\mu\}\{q_\nu\}} \equiv Y_{\{p_\mu\}\{q_\nu\}}(Z_{\nu\mu})$ are functions of the matrix elements $Z_{\nu\mu}$ of the matrix

$$\mathbf{Z} = \begin{pmatrix} \mathbf{I} - \mathbf{T}^+\mathbf{T} & -\mathbf{T}^+\mathbf{A} \\ -\mathbf{A}^+\mathbf{T} & \mathbf{I} - \mathbf{A}^+\mathbf{A} \end{pmatrix}. \tag{79}$$

Finally let us consider the transformation of coherent states.

$$|\psi_{\rm in}\rangle = |\gamma_1;\gamma_2,\gamma_3;\gamma_4\rangle$$
$$= \prod_{\nu=1}^{4} \exp\Big(\gamma_\nu \hat{\alpha}_\nu^\dagger - \gamma_\nu^* \hat{\alpha}_\nu\Big)\,|0\rangle. \tag{80}$$

From Eq. (59) we again obtain Eq. (65), where $|\psi_{\rm out}\rangle$ is the coherent state

$$|\psi_{\rm out}\rangle = |\lambda_1;\lambda_2,\lambda_3;\lambda_4\rangle$$
$$= \prod_{\mu=1}^{4} \exp\Big(\lambda_\mu \hat{\alpha}_\mu^\dagger - \lambda_\mu^* \hat{\alpha}_\mu\Big)\,|0\rangle, \tag{81}$$

with

$$\lambda_\mu = \sum_{\nu=1}^{4} \Lambda_{\mu\nu}\gamma_\nu\,. \tag{82}$$

From Eqs. (65) and (82) it follows that the outgoing modes are prepared in coherent states,

$$\hat{\varrho}_{\rm out}^{(\rm F)} = |\lambda_1;\lambda_2\rangle\langle\lambda_1;\lambda_2|. \tag{83}$$

Note that when the device is excited in a coherent state, then the coherent amplitudes λ_1 and λ_2 of the outgoing modes are not only determined by the characteristic transformation matrix $\mathbf{T}$ but also by the absorption matrix $\mathbf{A}$ and the coherent-state amplitudes of the device, as it can be seen from Eq. (82),

$$\lambda_i = \sum_{j=1}^{2} \left(T_{ij}\gamma_j + A_{ij}\gamma_{j+2}\right) \tag{84}$$

$(i=1,2)$.

IV. SUMMARY

We have developed a quantum theory of the action of a dispersive and absorbing optical four-port device, such as a beam splitter. In particular we have presented formulas for calculating the complete quantum state of the outgoing fields from the input quantum state of the incoming fields and the device excitations, without any frequency restriction. The theory is a natural extension of the standard theory of lossless beam splitters. According to the underlying quantization scheme for radiation in inhomogeneous Kramers–Kronig media, the

device is described in terms of a frequency-dependent transformation matrix that includes transmission and reflection and a frequency-dependent absorption matrix.

For each frequency the action of the device has been described in terms of a U(4) group transformation of incoming field operators and device operators. Each U(4) group transformation can be realized in a natural way by the combined action of eight lossless four-port devices. However, each U(4) matrix is only determined up to a U(2) matrix. This matrix can be chosen such that the U(4) group transformation is equivalent to five U(2) group transformations. That is to say, for chosen frequency the action of an absorbing four-port device formally corresponds to the combined action of five lossless four-port devices.

The quantum state of the outgoing radiation can be expected to sensitively depend on the quantum state the device is prepared in when the incoming fields impinge on the device. In combination with conditional measurement this offers novel possibilities of quantum state manipulation. In particular, the theory enables one to study the effect of resonance frequencies on quantum state transformation.

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APPENDIX A: DERIVATION OF THE U(4) GROUP MATRIX

Let us write the sought U(4) matrix $\mathbf{\Lambda}(\omega)$ as

$$\mathbf{\Lambda}(\omega) = \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{A}(\omega) \\ \mathbf{F}(\omega) & \mathbf{G}(\omega) \end{pmatrix}, \tag{A1}$$

where $\mathbf{T}(\omega)$ and $\mathbf{A}(\omega)$ are defined in Eqs. (14) and (15) and satisfy the relation (17). The 2×2 matrices $\mathbf{F}(\omega)$ and $\mathbf{G}(\omega)$ are to be determined such that $\mathbf{\Lambda}(\omega)$ is unitary, i.e.,

$$\mathbf{F}(\omega)\mathbf{F}^{+}(\omega) + \mathbf{G}(\omega)\mathbf{G}^{+}(\omega) = \mathbf{I}, \tag{A2}$$

$$\mathbf{F}(\omega)\mathbf{T}^{+}(\omega) + \mathbf{G}(\omega)\mathbf{A}^{+}(\omega) = \mathbf{0}\,. \tag{A3}$$

From Eq. (A3) we find that

$$\mathbf{F}(\omega) = -\mathbf{G}(\omega)\mathbf{A}^{+}(\omega)(\mathbf{T}^{+})^{-1}(\omega). \tag{A4}$$

We substitute in Eq. (A2) for $\mathbf{F}(\omega)$ the result of Eq. (A4) and derive

$$\mathbf{G}(\omega)\left\{\mathbf{I} + \mathbf{A}^{+}(\omega)\left[\mathbf{T}(\omega)\mathbf{T}^{+}(\omega)\right]^{-1}\mathbf{A}(\omega)\right\}\mathbf{G}^{+}(\omega) = \mathbf{I}, \tag{A5}$$

and hence

$$\mathbf{I} + \mathbf{A}^{+}(\omega)\left[\mathbf{T}(\omega)\mathbf{T}^{+}(\omega)\right]^{-1}\mathbf{A}(\omega) = \left[\mathbf{G}^{+}(\omega)\mathbf{G}(\omega)\right]^{-1}. \tag{A6}$$

Recalling Eq. (17), from Eq. (A6) we find that

$$\mathbf{G}^{+}(\omega)\mathbf{G}(\omega) = \mathbf{I} - \mathbf{A}^{+}(\omega)\mathbf{A}(\omega)\,. \tag{A7}$$

A particular solution of Eq. (A7) is

$$\mathbf{G}(\omega) = \mathbf{C}(\omega)\mathbf{S}^{\mathbf{-1}}(\omega)\mathbf{A}(\omega), \tag{A8}$$

where $\mathbf{C}(\omega)$ and $\mathbf{S}(\omega)$ are defined in Eqs. (32) and (33), respectively. Obviously, the general solution reads as

$$\mathbf{G}(\omega) = \mathbf{D}(\omega)\mathbf{C}(\omega)\mathbf{S}^{\mathbf{-1}}(\omega)\mathbf{A}(\omega), \tag{A9}$$

where $\mathbf{D}$ is an arbitrary unitary 2×2 matrix. From Eq. (A4) it then follows that $\mathbf{F}(\omega)$ is given by

$$\mathbf{F}(\omega) = -\mathbf{D}(\omega)\mathbf{S}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega). \tag{A10}$$

Combining Eqs. (A1), (A9), and (A10), we obtain

$$\mathbf{\Lambda}(\omega) = \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{A}(\omega) \\ -\mathbf{D}(\omega)\mathbf{S}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{D}(\omega)\mathbf{C}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix}, \tag{A11}$$

which reveals that for given matrices $\mathbf{T}(\omega)$ and $\mathbf{A}(\omega)$ the U(4) matrix $\mathbf{\Lambda}(\omega)$ is only determined up to a U(2) matrix $\mathbf{D}(\omega)$.

APPENDIX B: FACTORIZATION OF THE U(4) GROUP TRANSFORMATION

The U(4) matrix $\mathbf{\Lambda}(\omega)$ in Eq. (A11) can be rewritten as a product of three U(4) matrices as follows:

$$\mathbf{\Lambda}(\omega) = \mathbf{\Lambda}_3(\omega)\,\mathbf{\Lambda}_2(\omega)\,\mathbf{\Lambda}_1(\omega), \tag{B1}$$

where

$$\mathbf{\Lambda}_1(\omega) = \begin{pmatrix} \mathbf{D}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix}, \tag{B2}$$

$$\mathbf{\Lambda}_2(\omega) = \begin{pmatrix} \mathbf{D}(\omega)\mathbf{C}(\omega)\mathbf{D}^{+}(\omega) & \mathbf{D}(\omega)\mathbf{S}(\omega)\mathbf{D}^{+}(\omega) \\ -\mathbf{D}(\omega)\mathbf{S}(\omega)\mathbf{D}^{+}(\omega) & \mathbf{D}(\omega)\mathbf{C}(\omega)\mathbf{D}^{+}(\omega) \end{pmatrix}, \tag{B3}$$

$$\mathbf{\Lambda}_3(\omega) = \begin{pmatrix} \mathbf{D}^{\dagger}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{B4}$$

When we choose the matrix $\mathbf{D}(\omega)$ such that the matrices $\mathbf{D}(\omega)\mathbf{C}(\omega)\mathbf{D}^{+}(\omega)$ and $\mathbf{D}(\omega)\mathbf{S}(\omega)\mathbf{D}^{+}(\omega)$ become diagonal matrices [note that $\mathbf{C}(\omega)$ and $\mathbf{S}(\omega)$ defined in Eqs. (32) and (33), respectively, can be diagonalized by the same unitary matrix], then the U(4) group transformation corresponds to five U(2) group transformations.

Let $\mathbf{D}(\omega)$ be the unit matrix, $\mathbf{D}(\omega)=\mathbf{I}$. In this case Eq. (B1) reduces to

$$\mathbf{\Lambda}(\omega) = \mathbf{\Lambda}_2(\omega)\,\mathbf{\Lambda}_1(\omega), \tag{B5}$$

where now

$$\mathbf{\Lambda}_1(\omega) = \begin{pmatrix} \mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix} \tag{B6}$$

and

$$\mathbf{\Lambda}_2(\omega) = \begin{pmatrix} \mathbf{C}(\omega) & \mathbf{S}(\omega) \\ \\ -\mathbf{S}(\omega) & \mathbf{C}(\omega) \end{pmatrix}. \tag{B7}$$

The matrix $\mathbf{\Lambda}_2(\omega)$ can be given by the unitary transform of a quasi-diagonal matrix $\mathbf{\Lambda}'_2(\omega)$,

$$\mathbf{\Lambda}_2(\omega) = \mathbf{\Upsilon}^{+}\mathbf{\Lambda}'_2(\omega)\mathbf{\Upsilon}, \tag{B8}$$

where

$$\mathbf{\Lambda}'_2(\omega) = \begin{pmatrix} \mathbf{C}(\omega) - i\mathbf{S}(\omega) & \mathbf{0} \\ \\ \mathbf{0} & \mathbf{C}(\omega) + i\mathbf{S}(\omega) \end{pmatrix} \tag{B9}$$

and

$$\mathbf{\Upsilon} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & i\mathbf{I} \\ \\ i\mathbf{I} & \mathbf{I} \end{pmatrix}. \tag{B10}$$

Combining Eqs. (B5) and (B8), we obtain

$$\mathbf{\Lambda}(\omega) = \mathbf{\Upsilon}^{+}\mathbf{\Lambda}'_2(\omega)\mathbf{\Upsilon}\mathbf{\Lambda}_1(\omega), \tag{B11}$$

which corresponds to a decomposition of the U(4) group transformation into eight U(2) group transformations.

Using Eqs. (29) and (B5) and recalling Eqs. (35) – (37), we may write

$$\begin{aligned} \hat{\boldsymbol{\beta}}(\omega) &= \mathbf{\Lambda}_2(\omega)\,\mathbf{\Lambda}_1(\omega)\,\hat{\boldsymbol{\alpha}}(\omega) \\ &= \mathbf{\Lambda}_2(\omega)\,\hat{U}_1^{\dagger}\hat{\boldsymbol{\alpha}}(\omega)\hat{U}_1 \\ &= \hat{U}_1^{\dagger}\,\mathbf{\Lambda}_2(\omega)\,\hat{\boldsymbol{\alpha}}(\omega)\,\hat{U}_1 \\ &= \hat{U}_1^{\dagger}\,\hat{U}_2^{\dagger}\,\hat{\boldsymbol{\alpha}}(\omega)\,\hat{U}_2\hat{U}_1 \;=\; \hat{U}^{\dagger}\,\hat{\boldsymbol{\alpha}}(\omega)\,\hat{U}, \end{aligned} \tag{B12}$$

with

$$\hat{U} \equiv \hat{U}[\mathbf{\Lambda};\hat{\boldsymbol{\alpha}}] = \hat{U}[\mathbf{\Lambda}_2;\hat{\boldsymbol{\alpha}}]\hat{U}[\mathbf{\Lambda}_1;\hat{\boldsymbol{\alpha}}]. \tag{B13}$$

Here, $\hat{U}_i \equiv \hat{U}[\mathbf{\Lambda}_i;\hat{\boldsymbol{\alpha}}]$ $(i=1,2)$ is given by Eq. (36), with $\mathbf{\Phi}_i(\omega)$ in place of $\mathbf{\Phi}(\omega)$, and

$$\exp[-i\mathbf{\Phi}_i(\omega)] = \mathbf{\Lambda}_i(\omega). \tag{B14}$$

From the quasi-diagonal structure of $\mathbf{\Lambda}_1(\omega)$, Eq. (B6), it then follows that

$$\mathbf{\Phi}_1(\omega) = \begin{pmatrix} \mathbf{W}_1(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2(\omega) \end{pmatrix}, \tag{B15}$$

where

$$\exp[-i\mathbf{W}_1(\omega)] = \mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) \tag{B16}$$

and

$$\exp[-i\mathbf{W}_2(\omega)] = \mathbf{S}^{-1}(\omega)\mathbf{A}(\omega). \tag{B17}$$

Thus, $\hat{U}[\mathbf{\Lambda}_1; \hat{\boldsymbol{\alpha}}]$ can be expressed in terms of two unitary operators of the type given in Eq. (46) [together with Eq. (47)],

$$\hat{U}[\mathbf{\Lambda}_1; \hat{\boldsymbol{\alpha}}] = \hat{U}[\mathbf{S}^{-1}\mathbf{T}; \hat{\mathbf{g}}]\,\hat{U}[\mathbf{C}^{-1}\mathbf{T}; \hat{\mathbf{a}}]. \tag{B18}$$

To decompose $\hat{U}[\mathbf{\Lambda}_2; \hat{\boldsymbol{\alpha}}]$, we note that Eq. (B8) implies that

$$\hat{U}[\mathbf{\Lambda}_2; \hat{\boldsymbol{\alpha}}] = \hat{U}[\mathbf{\Lambda}_2'; \mathbf{\Upsilon}(\omega)\hat{\boldsymbol{\alpha}}], \tag{B19}$$

where

$$\exp[-i\mathbf{\Phi}_2'(\omega)] = \mathbf{\Lambda}_2'(\omega) \tag{B20}$$

and

$$\mathbf{\Upsilon}(\omega)\hat{\boldsymbol{\alpha}}(\omega) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{\mathbf{a}}(\omega) + i\hat{\mathbf{g}}(\omega) \\ i\hat{\mathbf{a}}(\omega) + \hat{\mathbf{g}}(\omega) \end{pmatrix}. \tag{B21}$$

The quasi-diagonal structure of $\mathbf{\Lambda}_2'(\omega)$, Eq. (B9), enables us to write

$$\mathbf{\Phi}_2'(\omega) = \begin{pmatrix} \mathbf{W}_3(\omega) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{W}_3(\omega) \end{pmatrix}, \tag{B22}$$

where

$$\exp[-i\mathbf{W}_3(\omega)] = \mathbf{C}(\omega) - i\mathbf{S}(\omega), \tag{B23}$$

so that $\hat{U}[\mathbf{\Lambda}_2; \hat{\boldsymbol{\alpha}}]$ can also be expressed in terms of two unitary operators of the type given in Eq. (46) [together with Eq. (47)],

$$\begin{aligned} \hat{U}[\mathbf{\Lambda}_2; \hat{\boldsymbol{\alpha}}] = \hat{U}[&\mathbf{C}+i\mathbf{S}; (i\hat{\mathbf{a}}+\hat{\mathbf{g}})/\sqrt{2}] \\ &\times \hat{U}[\mathbf{C}-i\mathbf{S}; (\hat{\mathbf{a}}+i\hat{\mathbf{g}})/\sqrt{2}]. \end{aligned} \tag{B24}$$

Recalling the definitions of $\hat{\mathbf{d}}_j(\omega)$, Eq. (51), and $\mathbf{P}$, Eq. (52), it is seen that $(\hat{\mathbf{a}}+i\hat{\mathbf{g}})/\sqrt{2}$ and $(i\hat{\mathbf{a}}+\hat{\mathbf{g}})/\sqrt{2}$ can be given by

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{a}_j(\omega) + i\hat{g}_j(\omega) \\ i\hat{a}_j(\omega) + \hat{g}_j(\omega) \end{pmatrix} &= \mathbf{P}\,\hat{\mathbf{d}}_j(\omega) \\ &= \hat{U}^\dagger[\mathbf{P}; \hat{\mathbf{d}}_j]\,\hat{\mathbf{d}}_j(\omega)\,\hat{U}[\mathbf{P}; \hat{\mathbf{d}}_j] \end{aligned} \tag{B25}$$

$(j=1,2)$, and hence

$$\begin{aligned} \hat{U}[\mathbf{C}-i\mathbf{S}; (\hat{\mathbf{a}}+i\hat{\mathbf{g}})/\sqrt{2}] &= \hat{U}^\dagger[\mathbf{P}; \hat{\mathbf{d}}_1]\,\hat{U}^\dagger[\mathbf{P}; \hat{\mathbf{d}}_2] \\ &\times \hat{U}[\mathbf{C}-i\mathbf{S}; \hat{\mathbf{a}}]\,\hat{U}[\mathbf{P}; \hat{\mathbf{d}}_2]\,\hat{U}[\mathbf{P}; \hat{\mathbf{d}}_1], \end{aligned} \tag{B26}$$

$$\begin{aligned} \hat{U}[\mathbf{C}+i\mathbf{S}; (i\hat{\mathbf{a}}+\hat{\mathbf{g}})/\sqrt{2}] &= \hat{U}^\dagger[\mathbf{P}; \hat{\mathbf{d}}_1]\,\hat{U}^\dagger[\mathbf{P}; \hat{\mathbf{d}}_2] \\ &\times \hat{U}[\mathbf{C}-i\mathbf{S}; \hat{\mathbf{g}}]\,\hat{U}[\mathbf{P}; \hat{\mathbf{d}}_2]\,\hat{U}[\mathbf{P}; \hat{\mathbf{d}}_1]. \end{aligned} \tag{B27}$$

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